Exercise with Solutions: Conditional vs unconditional expectations in RE models

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Exercise: (uncoupled) model with Rational Expectations

Consider a simple model described by the following three equations:

$$y_t = \beta \mathbb{E}_t y_{t+1} + x_t \tag{1}$$

$$x_t = \phi + \rho x_{t-1} + \varepsilon_t \ \ , \quad \varepsilon_t \sim \mathcal{N}\left(0, \sigma^2\right) \tag{2}$$

$$z_t = \sigma + \mu y_t \tag{3}$$

where $\{\beta, \phi, \rho, \sigma, \mu\}$ are parameters.

1. *What kind of variables (forward-looking, backward-looking, and static) do we have in this model?*
2. *To secure one stable solution for this model, what are the constraints that we have to impose upon the parameters?*
3. *Solve for the model's deterministic steady-state (or long-term equilibrium).*

4. *Given the following parameters, what are the long-term equilibrium levels of y_t, x_t and z_t, according to the hypothesis of **conditional expectations**?*

$$\beta = 0.75\ ,\ \phi = 10\ ,\ \rho = 0.5\ ,\ \sigma = 2\ ,\ \mu = 0.1$$

5. *Now consider that the system is in its long-term equilibrium. If in a given period t, x_t suffers a shock equal to $\varepsilon_t = +1$ (no more shocks afterward), what happens to x_t, y_t, and z_t? And what will their values be in $t+1$?*

6. *Considering the same shock and the same parameters as above, what happens to y_t, x_t, and z_t, according to the hypothesis of **unconditional expectations**?*

7. *When will the two solutions (under conditional and unconditional expectations) be the same?*

Solutions

1. *What kind of variables (forward-looking, backward-looking, and static) do we have in this model?*

Solution 1.

This is a dynamic model and has three variables: y_t (a forward-looking variable); x_t (a backward-looking variable also called as predetermined), and z_t which is a static variable (no dynamics in itself).

2. *To secure one stable solution, what are the constraints that we have to impose upon the parameters?*

Solution 2.

As we have a dynamic model, its stability depends on the eigenvalues associated with the system. As this model has uncoupled blocks (each equation can be solved separately), the eigenvalues correspond to individual parameters: β in eq. (1) and ρ in eq. (2). Eq. (3) represents a static variable (z_t), and static variables have no internal dynamics, and so the parameters in eq. (3) are irrelevant to the stability of this model.

Therefore, to secure a unique and stable equilibrium in this model, we have to impose the following conditions:

$$|\beta| < 1 \, , \; |\rho| < 1$$

3. *Solve for the model's deterministic steady-state (or long-term equilibrium).*

Solution 3.

The deterministic solution of the model is the solution without the random component ($\varepsilon_t = 0, \forall t$). It is as if no random factors were affecting the model's dynamics. Therefore, to solve for the deterministic steady state, we have to impose the following conditions:

$$x_t = x_{t-1} = \overline{x}\ ,\ y_t = \mathbb{E}_t y_{t+1} = \overline{y}\ ,\ z_t = z_{t-1} = \overline{z}$$

Applying those conditions to each of the three equations above, we get:

Solution 3 (continuation).

$$\bar{x} = \phi + \rho\bar{x} + 0 \;\Rightarrow\; \bar{x}(1-\rho) = \phi \;\Rightarrow\; \bar{x} = \frac{\phi}{1-\rho} \tag{4}$$

$$\bar{y} = \beta\bar{y} + \bar{x} \;\Rightarrow\; \bar{y}(1-\beta) = \bar{x} \;\Rightarrow\; \bar{y} = \frac{\bar{x}}{1-\beta} = \frac{\phi/(1-\rho)}{1-\beta} \tag{5}$$

$$\bar{z} = \sigma + \rho\bar{y} \;\Rightarrow\; \bar{z} = \sigma + \frac{\mu\phi/(1-\rho)}{1-\beta} \tag{6}$$

4. *Given the following parameters, what are the long-term equilibrium levels of y_t, x_t and z_t, according to the hypothesis of **conditional expectations**?*

$$\beta = 0.75 \ , \ \phi = 10 \ , \ \rho = 0.5 \ , \ \sigma = 2 \ , \ \mu = 0.1$$

Solution 4.

Let us start with eq. (2). Iterate backward in time up to the 3rd iteration, then generalize to the n-th iteration. Assuming that $|\rho| < 1$ in order to rule out explosive behavior, we get (jump to Appendix 1 for the derivation details):

Solution 4 (continuation).

$$x_t = \phi + \rho x_{t-1} + \varepsilon_t \;\Rightarrow\; x_t = \underbrace{\frac{\phi}{1-\rho}}_{deterministic} + \underbrace{\sum_{i=0}^{n-1} \rho^i \varepsilon_{t-i}}_{random\ part} \tag{7}$$

If $\phi = 10$, $\rho = 0.5$, then

$$x_t = \underbrace{\frac{10}{1-0.5}}_{determinist\ part} + \underbrace{\sum_{i=0}^{n-1} 0.5^i \varepsilon_{t-i}}_{random\ part} = \underbrace{20}_{\bar{x}} + \underbrace{\sum_{i=0}^{n-1} 0.5^i \varepsilon_{t-i}}_{x^{\varepsilon}} \tag{8}$$

Therefore, we can simply write the solution to x_t as the sum of a deterministic part $(\bar{x})$ and a a random component (x^{ε}):

$$x_t = \bar{x} + x^{\varepsilon} \tag{8a}$$

Solution 4 (continuation).

It is easy to see that the solution in (8)-(8a) depends on the shocks that may hit this process over time.

- If we have no shocks, $x^{\varepsilon} = 0$, and we are back to the deterministic case where $x_t = \bar{x} = 20$
- If we have one shock, we will see what happens in question 5
- If we have many shocks, it is better to use the computer

Now, let us move on to eq. (1). Iterate (now) forward up to the 3rd iteration, then generalize to the n-th iteration. Assuming that $\beta < |1|$ in order to rule out explosive behavior, we get (jump to Appendix 2 for the derivation details):

$$y_t = \beta \mathbb{E}_t y_{t+1} + x_t \quad \Rightarrow \quad y_t = \sum_{i=0}^{n-1} \beta^i \mathbb{E}_t x_{t+i} \tag{9}$$

But, in eq. (9), what is the value of $\mathbb{E}_t x_{t+i}$? Considering that in eq. (2) we have $x_t = \phi + \rho x_{t-1} + \varepsilon_t$, then we will get (jump to Appendix 3 for details of the result below, or see slide 18 in "6. The Simplest DSGE Model")

$$\mathbb{E}_t x_{t+i} = \underbrace{\frac{\phi}{1-\rho}}_{deterministic} + \underbrace{\rho^i x_t^{\varepsilon}}_{random} \tag{10}$$

Inserting eq. (10) into (9), we get:

$$y_t = \sum_{i=0}^{n-1} \beta^i \mathbb{E}_t x_{t+i} = \sum_{i=0}^{n-1} \beta^i \left(\frac{\phi}{1-\rho} + \rho^i x_t^\varepsilon \right) = \underbrace{\sum_{i=0}^{n-1} \beta^i \left(\frac{\phi}{1-\rho} \right)}_{deterministic} + \underbrace{\sum_{i=0}^{n-1} (\beta\rho)^i x_t^\varepsilon}_{random}$$

Now, we must obtain the solution to the two geometric sums above. Jump to Appendix 4 to get the details associated with a geometric sum and its solution, or see slide 8 in "5. Solving Rational Expectations Models". The result will come out as follows:

$$y_t = \underbrace{\frac{\phi/(1-\rho)}{1-\beta}}_{deterministic} + \underbrace{\frac{x_t^\varepsilon}{1-\beta\rho}}_{random} \tag{11}$$

Taking into account that $\phi = 10, \rho = 0.5, \beta = 0.75$, the solution to eq. (11) is given by:

$$y_t = \frac{10/(1-0.5)}{1-0.75} + \frac{x_t^{\varepsilon}}{1-0.75\times 0.5} = \underbrace{80}_{determ.} + \underbrace{1.6x_t^{\varepsilon}}_{random} \tag{12}$$

It is easy to see what forces affect the determination of y_t:

- The first one is the impact of the deterministic value of x_t, which is ($\overline{x} = 10/(1-0.5) = 20$). This impact is given by the following term in eq. (12):

$$\frac{10/(1-0.5)}{1-0.75} = 80$$

- The second one is the (indirect) impact that the shock exerts upon y_t, given by $1.6x_t^{\varepsilon}$. Graphically, this impact can be represented as: $\varepsilon_t \rightarrow \underbrace{x_t^{\varepsilon} \rightarrow y_t}_{1.6x_t^{\varepsilon}}$

- If there are no shocks, we will have $x_t^{\varepsilon} = 0$, and $y_t = \overline{y} = 80$. We are back to the deterministic solution.

5. *Now consider that the system is in its long-term equilibrium. If in a given period t,* x_t *suffers a shock equal to* $\varepsilon_t = +1$ *(no more shocks afterward), what happens to* x_t*,* y_t*, and* z_t*? And what will their values be at* $t+1$*?*

Solution 5

Here, we have to start with x_t because this variable directly suffers the shock. The dynamics of x_t is given by eq. (8) above. Let us bring it back:

$$x_t = \underbrace{20}_{\text{deterministic}} + \underbrace{\sum_{i=0}^{n-1} 0.5^i \varepsilon_{t-i}}_{\text{random part}} \tag{8a}$$

And the shock is this:

$$\varepsilon_t = +1 \; , \; \varepsilon_{t+i} = 0$$

If the shock occurs at period t, let us iterate the model forward two times: $i = 0, 1, 2$, and see what happens to our variable x over time (x_{t+i}). Notice that $t + 0$ is when the shock occurs (the initial period).

$$
\begin{aligned}
i = 0 \Rightarrow x_{t+0} &= 20 + 0.5^i \varepsilon_{t+0-i} = 20 + 0.5^0 \times \varepsilon_{t+0-0} = && 20 + && 0.5^0 \times 1 = 21 \\
i = 1 \Rightarrow x_{t+1} &= 20 + 0.5^i \varepsilon_{t+1-i} = 20 + 0.5^1 \times \varepsilon_{t+1-1} = && 20 + && 0.5^1 \times 1 = 20.5 \\
i = 2 \Rightarrow x_{t+2} &= 20 + 0.5^i \varepsilon_{t+2-i} = 20 + 0.5^2 \times \varepsilon_{t+2-2} = && \underbrace{20}_{deterministic} + && \underbrace{0.5^2 \times 1}_{random} = 20.25
\end{aligned}
$$

As expected, the deterministic part of x_t remains constant $(\overline{x} = 20)$. The change occurs in the random part of this process (x^{ε}_{t+i}):

$$
x_t = 20 + \underbrace{1}_{x^{\varepsilon}_{t}},\ x_{t+1} = 20 + \underbrace{0.5}_{x^{\varepsilon}_{t+1}},\ x_{t+2} = 20 + \underbrace{0.25}_{x^{\varepsilon}_{t+2}},\ldots
$$

As we know the relationship between y_t and x_t that solves the model, we can obtain the expected impact of such shock upon y_t. That relationship was given above by eq. (12). Let us bring it back:

$$y_t = \underbrace{80}_{\text{determ.}} + \underbrace{1.6 x_t^{\varepsilon}}_{\text{random}} \tag{12a}$$

As the deterministic part of x does not change, only its random part does, it is immediate to get that:

$$\begin{aligned} y_{t+0} &= 80 + 1.6 \times 1 &&= 81.6 \\ y_{t+1} &= 80 + 1.6 \times 0.5 &&= 80.8 \\ y_{t+2} &= 80 + 1.6 \times 0.25 &&= 80.4 \end{aligned}$$

The shock that hits x_t has a persistent impact over time, indirectly affecting y_t as well.

Now, consider the static variable (eq. 3):

$$z_t = 2 + 0.1 y_t$$

It is immediate to see the impact of the shock $\varepsilon_t = +1$ upon z_t:

$$z_{t+0} = 2 + 0.1 \times 81.6 = 10.16$$
$$z_{t+1} = 2 + 0.1 \times 80.8 = 10.08$$
$$z_{t+2} = 2 + 0.1 \times 80.4 = 10.04$$

6. *Considering the same shock and the same parameters as above, what happens to y_t, x_t, and z_t, according to the hypothesis of* ***unconditional expectations****?*

Solution 6

- With unconditional expectations, everything becomes easier to calculate.
- The solution to x_t will be the same (eq. 8):

$$x_t = \underbrace{20}_{\text{deterministic}} + \underbrace{\sum_{i=0}^{n-1} 0.5^i \varepsilon_{t-i}}_{\text{random part}} \tag{8b}$$

- The solution to y_t is different in the case of unconditional expectations because, in this case, we care about the unconditional mean of x in the solution to y_t. This solution was presented above (eq. 9). Let us bring it back:

Solution 6 (continuation)

$$y_t = \sum_{i=0}^{n-1} \beta^i \mathbb{E}_t x_{t+i} \tag{9b}$$

- What is the ***unconditional*** mean of x_t? It is $\mathbb{E}_t x_{t+i} = \overline{x} = 20$. Jump to Appendix 5 for further details. So, just insert this result into eq. (9b), and we will get:

$$y_t = \sum_{i=0}^{n-1} \beta^i \mathbb{E}_t x_{t+i} = \sum_{i=0}^{n-1} \beta^i \, \overline{x} = \frac{\overline{x}}{1-\beta} = \frac{20}{1-0.75} = 80 \tag{13}$$

- The solution for z_t is immediate: insert the result in eq. (13) into eq. (3):

$$z_t = \sigma + \mu y_t = \sigma + \frac{\mu \overline{x}}{1-\beta} = 2 + \frac{0.1 \times 20}{1-0.75} = 10$$

Solution 6 (continuation)

- Notice that the results in the two previous slides can be summarized as:

$$x_t = \overline{x} + \sum_{i=0}^{n-1} 0.5^i \varepsilon_{t-i} = 20 + \sum_{i=0}^{n-1} 0.5^i \varepsilon_{t-i}$$

$$y_t = 80 \ , \ z_t = 10 \ , \forall t$$

- The solutions to y_t and z_t are the same as those in the case of the deterministic solution in Question 3: see eqs. (5) and (6).
- But they are different from those obtained under ***conditional expectations***, because in the current case we do not take into account the impact from the shocks that affect x_t.

Solution 6 (continuation)

- The solution to x_t is the same under conditional and unconditional expectations because x_t is a predetermined (or backward-looking) variable.
- So, if shocks hit the economy, unconditional expectations will provide a wrong answer to what truly happens in the economy. That is why all modern macroeconomic models use conditional expectations.

7. *When will the two solutions (under conditional and unconditional expectations) be the same?*

Solution 7

Only in the following case: no shocks will hit the economy (model) over time.

Appendices

Each one includes important techniques that are required for solving the questions above.

Appendix 1. Solving a backward-looking equation with noise
(jump back)

$$
\begin{aligned}
(iteration\ 1) \rightarrow \quad x_t &= \phi + \rho x_{t-1} + \varepsilon_t \\
(iteration\ 2) \rightarrow \quad x_t &= \phi + \rho\left(\phi + \rho x_{t-2} + \varepsilon_{t-1}\right) + \varepsilon_t \\
&= \phi + \rho\phi + \rho^2 x_{t-2} + \rho\varepsilon_{t-1} + \varepsilon_t \\
(iteration\ 3) \rightarrow \quad x_t &= \phi + \rho\phi + \rho^2\left(\phi + \rho x_{t-3} + \varepsilon_{t-2}\right) + \rho\varepsilon_{t-1} + \varepsilon_t \\
&= \phi + \rho\phi + \rho^2\phi + \rho^3 x_{t-3} + \rho^2\varepsilon_{t-2} + \rho\varepsilon_{t-1} + \varepsilon_t \\
&= \sum_{i=0}^{3-1} \rho^i \phi + \rho^3 x_{t-3} + \sum_{i=0}^{3-1} \rho^i \varepsilon_{t-i} \\
(iteration\ n) \rightarrow \quad x_t &= \sum_{i=0}^{n-1} \rho^i \phi + \rho^n x_{t-n} + \sum_{i=0}^{n-1} \rho^i \varepsilon_{t-i} \\
&= \sum_{i=0}^{n-1} \rho^i \phi + \sum_{i=0}^{n-1} \rho^i \varepsilon_{t-i}\ , \quad \text{as} \quad \rho < |1| \Rightarrow \rho^n x_{t-n} \rightarrow 0.
\end{aligned}
$$

Appendix 2. Solving a forward-looking equation (jump back)

$$
\begin{aligned}
(iteration\ 1) \rightarrow \quad y_t &= \beta \mathbb{E}_t y_{t+1} + x_t \\
(iteration\ 2) \rightarrow \quad y_t &= \beta \left(\beta \mathbb{E}_t y_{t+2} + \mathbb{E}_t x_{t+1} \right) + x_t \\
&= \beta^2 \mathbb{E}_t y_{t+2} + \beta \mathbb{E}_t x_{t+1} + x_t \\
(iteration\ 3) \rightarrow \quad y_t &= \beta^2 \left(\beta \mathbb{E}_t y_{t+3} + \mathbb{E}_t x_{t+2} \right) + \beta \mathbb{E}_t x_{t+1} + x_t \\
&= \beta^3 \mathbb{E}_t y_{t+3} + \beta^2 \mathbb{E}_t x_{t+2} + \beta \mathbb{E}_t x_{t+1} + x_t \\
&= \beta^3 \mathbb{E}_t y_{t+3} + \sum_{i=0}^{3-1} \beta^i \mathbb{E}_t x_{t+i} \\
(iteration\ n) \rightarrow \quad y_t &= \beta^n \mathbb{E}_t y_{t+n} + \sum_{i=0}^{n-1} \beta^i \mathbb{E}_t x_{t+i} \\
&= \sum_{i=0}^{n-1} \beta^i \mathbb{E}_t x_{t+i} \ , \quad \text{as } |\beta| < 1 \Rightarrow \beta^n \mathbb{E}_t y_{t+n} \rightarrow 0.
\end{aligned}
$$

Appendix 3. Conditional expectations (jump back)

Suppose that x_t is given by the following stochastic process (noise affects x_t):

$$x_t = \phi + \rho x_{t-1} + \varepsilon_t \quad , \quad \varepsilon_t \sim \mathcal{N}\left(0, \sigma^2\right) \tag{A1}$$

To compute the conditional expectation of x_{t+i}, $(\mathbb{E}_t x_{t+i})$, apply the expectations operator to eq. (A1), up to the third iteration:

$$\begin{aligned}
x_t &= \phi + \rho x_{t-1} + \varepsilon_t \\
\mathbb{E}_t x_{t+1} &= \phi + \rho \mathbb{E}_t x_t + \mathbb{E}_t \varepsilon_{t+1} = \phi + \rho x_t + 0 &&= \phi + \rho x_t \\
\mathbb{E}_t x_{t+2} &= \phi + \rho \mathbb{E}_t x_{t+1} + \mathbb{E}_t \varepsilon_{t+2} = \phi + \rho\left[\phi + \rho x_t\right] + 0 &&= \phi + \rho\phi + \rho^2 x_t \\
\mathbb{E}_t x_{t+3} &= \phi + \rho \mathbb{E}_t x_{t+2} + \mathbb{E}_t \varepsilon_{t+3} = \phi + \rho\left[\phi + \rho\phi + \rho^2 x_t\right] + 0 &&= \underbrace{\phi + \rho\phi + \rho^2\phi}_{=\sum_{k=0}^{3-1} \phi\rho^k} + \rho^3 x_t
\end{aligned}$$

That is, at the 3rd iteration, we get:

$$\mathbb{E}_t x_{t+3} = \sum_{k=0}^{3-1} \phi\rho^k + \rho^3 x_t$$

Generalizing to the ith iteration, we obtain:

$$\mathbb{E}_t x_{t+i} = \sum_{k=0}^{i-1} \phi \rho^k + \rho^i x_t = \frac{\phi}{1-\rho} + \rho^i x_t \qquad \text{(A2)}$$

But as the deterministic mean of x_t is given by $\bar{x} = \frac{\phi}{1-\rho}$, then we can rewrite (A2) as:

$$\mathbb{E}_t x_{t+i} = \underbrace{\frac{\phi}{1-\rho}}_{\bar{x}} + \underbrace{\rho^i x_t}_{x^{\varepsilon}} = \bar{x} + \rho^i x_t^{\varepsilon} \qquad \text{(A3)}$$

where x^{ε} is the random component of this process affecting x_t. (jump back)

Appendix 4: Solution of a Geometric Series (jump back)

- Suppose we have a process that is written as:

$$s = \rho^0\phi + \rho^1\phi + \rho^2\phi + \rho^3\phi \; + \; \ldots \; = \sum_{i=0}^{\infty} \phi\rho^i$$

- It has two crucial elements:
  - First term of the series (when $i = 0$): ϕ
  - The common ratio: ρ
- The solution is given by the expression:

$$s = \frac{\text{first term}}{1 - \text{ common ratio}} = \frac{\phi}{1-\rho} \;, \qquad if \;\; |\rho| < 1$$

- No solution: if $|\rho| > 1$, s is explosive; if $|\rho| = 1$, s does not converge to any value.

Appendix 5. Unconditional expectations (jump back)

- Suppose that x_t is given by the following stochastic process:

$$x_t = \phi + \rho x_{t-1} + \varepsilon_t \quad , \qquad \varepsilon_t \sim \mathcal{N}\left(0, \sigma^2\right)$$

- Assuming unconditional expectations, the mean is given by the (deterministic) steady-state value of x_t:

$$x_t = x_{t-1} = \overline{x}$$

- Which leads to:

$$\overline{x} = \phi + \rho \overline{x} + 0 \Rightarrow \overline{x} = \frac{\phi}{1-\rho} \quad , \qquad \rho \neq 1.$$

- Therefore, the expected (unconditional) value of $\mathbb{E}_t x_{t+i}$ is given by:

$$\mathbb{E}_t x_{t+i} = \overline{x} = \frac{\phi}{1-\rho} \tag{A4}$$

Gladys' doubt

$$y_t = \underbrace{\sum_{i=0}^{n-1} \beta^i \left(\frac{\phi}{1-\rho} \right)}_{\text{deterministic}} + \underbrace{\sum_{i=0}^{n-1} (\beta\rho)^i x_t}_{\text{random}} \tag{10a}$$

We have two geometric sums in the eq. that precedes (11). Let us call it as eq. (10a). The solution to a geometric sum is:

$$s = \frac{first\ term}{1 - common\ ratio}$$

where the ***first term*** is the term of the sum when $i = 0$, and the ***common ratio*** is the constant that is raised to the power of i. Applying this principle to the two sums above, we get

$$y_t = \frac{\frac{\phi}{1-\rho}}{1-\beta} + \frac{x_t}{1-\beta\rho} \tag{11a}$$

That is all. Eq. (12) is the same as eq(11), but with the parameter values inserted.